DESIGNS FOR PAIRED AND TRIAD COMPARISONS

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INTRODUCTION

If $X=(x_1, x_2, x_3,\ldots\ldots x_m)$ be a set of $m>2$ distinct objects, a set of paired comparisons of X is the relation R in X which is antisymmetric and antireflexive; that is a subset of $X\times X$ such that $(x_i, x_i)\Sigma R$ and if $(x_i, x_j)\Sigma R$, then $(x_j, x_i)\Sigma R$. Such relation is called a comparison of X. The symbol $(x_i, x_j)\Sigma R$ means that in the comparison R, x_i is preferred to x_j, in symbols this may be expressed as $x_i>x_j$. The number of such comparisons in R is

$$K=\frac{m(m-1)}{2}.$$

A path K in R from y_1 to y_k and some times donoted by $(y_1, y_2,\ldots y_k)$ is a finite collection of ordered pairs

$$(y_1, y_2)\Sigma R, (y_2, y_3)\Sigma R,\ldots\ldots(y_k-1, y_k)\Sigma R.$$

If $y_1=y_k$, then the path is called circuit. The path is called elementary if all elements except y_1 and y_k are distinct. If k is a path in R with $y_1\neq y_k$, it is easy to see that there is an elementary path $K'\subset K$ from y_1 to y_k. However a circuit need not contain an elementary circuit with a given pair of elements.

PROBABILISTIC MODELS

For giving a theoretical foundation, probabilistic models can be introduced for the method of paired comparisons. One model was introduced by Bradley and Terry (1952) and another by Thurstone and Mosteller (1951). Briefly the basis of these models are given below.

It is postulated that associated with each of the m objects $x_1, x_2, \ldots x_m$ there exists parameters $\pi_i (i=1, 2 \ldots m)$ such that $\pi_i \geqslant 0$ and

$$\sum_{i=1}^{m} \pi_i = 1.$$

The parameters are further defined with the probability statement that

$$\left\{ P(x_i > x_j) \right\} = \frac{\pi_i}{\pi_i + \pi_j}$$

in the comparison of x_i with x_j. Maximum likelihood estimates of π_i and the formula for likelihood ratio tests have been given by Bradley and Terry (1952). Tables for small sample sizes and small number of treatments have been given by them In this analysis it is assumed that each pair is given to an equal number of judges. Similar methods have been developed by Dykstra (1960) when each pair is repeated an unequal number of times as in a balanced incomplete block design. Thompson W.A. and Russel Remage Junior (1964) have discussed the problem of weak stochastic ranking from paired comparisons without assuming an intrinsic worth for each x_i.

In the model by Bradley and Terry (1952), the probability of the observed result in n_{ij} repetitions on the comparison of the objects i and j is

$$\left(\frac{\pi_i}{\pi_i + \pi_j}\right)^{a_{ij}} \left(\frac{\pi_j}{\pi_i + \pi_j}\right)^{a_{ij}} \quad \ldots (1)$$

where

$$a_{ij} = 2n_{ij} - \sum_{k=1}^{n_{ij}} r_{ijk}, \quad a_{ij} + a_{ji} = n_{ij} \quad \ldots (2)$$

where $r_{ijk} = 1$ if $X_i > X_j$ and $r_{ijk} = 2$ if $X_j > X_i$ in the Kth repetition of the pairs (i, j). Here a_{ij} is the number of times the ith treatment ranks ahead of the jth treatment. From this the general likelihood function $L(\pi_i)$ is obtained by multiplying the appropriate expressions for all repetitions of all pairs as

$$L(\pi_i) = \prod_{i} \pi_i a^i \prod_{i<j} (\pi_i + \pi_j)^{-n_{ij}} \quad \ldots (3)$$

where

$$a_i = \sum_{j \neq i} a_{ij} \quad \ldots (4)$$

In (4) the index of summation j takes all values except i. When all the n_j's are equal to n the corresponding equation of Bradley and Terry would be generated. In the above and subsequent likelihood functions stochastic independence between pairs of treatments and repetitions are assumed. Maximising the likelihood by using appropriate Lagrange multipliers we get the estimate of p_i as

$$p_i = a_i / [\sum_{j \neq i} n_{ij} / (p_i + p_j)] \qquad ...(5)$$

An initial estimate of p_i is obtained by any method and by substituting on the right hand side of (5) and by iteration, good estimate of p_i can be obtained. To test the null hypothesis

$$H_0 : \pi_i = 1/t, \; i = 1, \ldots\ldots t$$

against the Bradley and Terry (1952) alternative hypothesis,

H_1 ; no π_i is assumed equal to any $\pi_j (i \neq j)$, the likelihood ratio test yields the statistic

$$B_1 = \sum_{i<j} n_{ij} \log (p_i + p_j) - \sum_i a_i \log p_i$$

and letting λ to be the likelihood ratio

$$-2 \log_e \lambda = 2(\sum_{i<j} n_{ij}) \log_e 2 - 2B_1 \log_e 10$$

is distributed in the limit as a x^2 with $t-1$ degrees of freedom when all n_{ij} become large.

Modified Thurstone-Mosteller Model

A modified Thurstone-Mosteller model has been developed by Glen and David (1960) which takes into account ties in paired comparisons as well. The paired comparison experiment was introduced by Thurstone for the purpose of estimating the relative strengths of treatment stimulus through subjective testing. He postulates a subjective continuum on which sensations are jointly normally distributed with equal standard deviations and zero correlations between pairs. Without further loss of generality we may have the scale of sensation continuum be so chosen as the difference of the true stimulus responses of a pair of treatments. Under this model the probability distribution of the difference of the two responses is normal with mean ∂ and unit variance.

Thurstone-Mosteller model prohibits the declaration of ties. If the difference between two responses lies below a certain threshold, say between $-J$ and $+J$, the judge will declare a tie. If π_t is the probability that a tie will be declared, in a paired comparison experiment involving t treatments and let x_i and x_j be single responses of a judge to the ith and jth stimuli. Let S_i denote the true response of the ith stimulus ($i=1,\ldots\ldots t$). Under the Thurstone-Mosteller model, the probability distribution of the difference $X_i - X_j (i \neq j)$ is normal with mean $S_i - S_j$ and unit variance. Define

$$F(a) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{\infty} e^{-y^2/2} \, d_y$$

from which we find

$$F(-a) = 1 - F(a).$$

Denoting treatments i and j by T_i and T_j respectively, we may write the probalility that T_i is preferred when T_i and T_j are compared as

$$\pi_{ij} = P(X_i > X_j) = F(S_i - S_j).$$

Under the modification of Thurstone Mosteller model we define the parameters

$$\begin{aligned} \pi_{i \cdot ij} &= P[(X_i - X_j) > T] = F(-T + S_i - S_j) \\ \pi_{j \cdot ij} &= P[(X_i - X_j) < -T] = 1 - F(T + S_i - S_j) \\ \pi_{o \cdot ij} &= P[\,|X_i - X_j| \leqslant T] = F(T + S_i - S_j) \\ &\qquad - F(-T + S_i - S_j) \end{aligned}$$

which are in turn the probalitity that T_i, T_j or neither are preferred in the comparison of T_i and T_j. Thus we get the relations.

$$\begin{aligned} \pi_{i \cdot ij} + \pi_{o \cdot ij} &= F(T + S_i - S_j) \ \& \\ \pi_{j \cdot ij} + \pi_{o \cdot ij} &= F(T - S_i + S_j). \end{aligned}$$

Suppose n observations are made in the comparison of T_i and T_j either by a single judge or by a group of judges having equal discriminatory powers relative to the stimuli concerned. Let the data recorded be

$p_{i \cdot ij} = n_{i \cdot ij}/n =$ proportion of preferences for T_i ;

$p_{j \cdot ij} = n_{j \cdot ij}/n =$ proportion of preferences for T_j ; and

$p_{o \cdot ij} = n_{o \cdot ij}/n =$ proportion of ties, when T_i and T_j

are compared where

$$n_{i \cdot ij} + n_{j \cdot ij} + n_{o \cdot ij} = n.$$

Under assumption of independence of these proportions and replacing $F(a)$ by the function

$$\frac{1}{2} \int_{-a}^{\pi/2} \cos y \, dy = \frac{1}{2}(1 + \sin a)$$

where a represents the angle in radian measure

$$-\frac{\pi}{2} \leqslant a \leqslant \frac{\pi}{2}$$

we find that

$$J_{(ij)}' = \tfrac{1}{2}[\sin^{-1}(2a_{ij}-1)+\sin^{-1}(2a_{ji}-1)] \text{ and}$$
$$S_i'-S_j' = \tfrac{1}{2}[\sin^{-1}(2a_{ij}-1)-\sin^{-1}(2a_{ji}-1)]$$

where $J_{(ij)}'$ and $S_i'-S_j'$

are experimental values of T and $S_i - S_j$. For large samples it can be shown $J_{(ij)}'$ and $(S_i'-S_j')$ are uncorrelated. Again, for large samples the correlation between $\sin^{-1}(2a_{ij}-1)$ and $\sin^{-1}(2a_{ji}-1)$ can be shown to be

$$\rho_{ij} = -\sqrt{\frac{\pi_{i\ ij}\pi_{j\ ij}}{(1-\pi_{i\ ij})(1-\pi_{j\ ij})}}$$

and hence that

$$\text{Var } [T_{ij}'] = (1+\rho_{ij})/2n \text{ and}$$
$$\text{Var } (S_i'-S_j'] = (1-\rho_{ij})/2n.$$

For $\pi_{o.ij} \neq 0$ these variances will not in general be homogeneous over all pairs $(i, j=1, 2......t)$. However in the absence of extreme comparisons the departures from homogeneity will be relatively small. The magnitude of the parameter T evidently depends on the ability of the judge or judges to detect small differences in the stimuli. When the judges may be regarded as equally competent, it is reasonable to assume a common value for T. The stimuli may be compared by comparing $S_i'-S_j'$ with the appropriate standard error.

In the triad comparison n objects are compared 3 at a time. The duo-trio and triangular tests involve matching without ranking whereas in the triad comparisons ranking is involved. It has been suggested that the triangular test is most efficient, but experimental evidence to the contrary has been reported (Hopkins and Gridgeman 1955) The number of triads that can be formed out of n objects is $n(n-1)(n-2)/3$! In using paired comparison for flavour intensity testing a unit trial consists in submitting coded preparations of the varieties in question tô a judge in the sequence (X, Y) or (Y, X) and requiring to rank them for palatability. The duo-trio test consists in submitting identified X or Y with the coded sequence (X), (Y) or (Y), (X) and requiring the subject to match the identified with the like coded variety. In the triangular test one of the completely coded sequences (X), (X),

(Y) ; (X), (Y), (X) ; Y, X, X ; Y, Y, X ; Y, X, Y ; *or* (X), (Y), (Y) is offered to the subject, again requiring an attempted marking of like aliquots. Inferences respecting the occurrence or non-occurrence of real discrimination are then made by relating the actual frequency of ranking or marking in repeated trials to percentage points of the binomial distribution expected in the absence of discrimination. Triad comparison is a variant of triangular test. Models similar to those by Bradley and Terry and Thurstone-Mosteller can be developed for triads as well.

Fractional Pairs And Fractional Triads

When the number of objects to be compared is very large paired and triad comparisons will be very taxing to the judges. Hence we propose fractional pairs and fractional triads and detail the split up as given below. The pairs that can be formed out of four objects *A, B, C, D* are *AB, BC, AC, AD, BD, CD*. This group can be split up into two sub-groups, those containing a specified object and those not containing that object. Thus the split up is (*AB, BC, AC*) and (*AD, BD, CD*). From the first group preferences of *D* with the objects cannot be obtained, but from the second preferences for all the objects can be obtained. Hence the comparisons can be made by randomising the second group properly and by giving each pair to each judge an equal number of times. Thus if each pair is given to each judge an even number n of times, the pair *A, D* can be offered on $n/2$ times in the order *AD* and $\frac{n}{2}$ times the order *DA*. We can try different types of designs for the problem. Let us examine the triangular design. The number of treatments in the design is $t = n'(n'-1)/2$ where n' is the side of the square. For $t=3$ and $n'=3$, the design would be as follows :—

X	1	2
1	X	3
2	3	X

We are here forming a square of size 3; delete the diagonal and fill in the treatment numbers symmetrically around the diagonal. Here the treatments will correspond to pairs. The designs of Clatworthy can be used also for selection of pairs. Thus if we have 10 varieties a triangular design with $t=10$ and $n'=5$ can be used.

X	1	2	3	4
1	X	5	6	7
2	5	X	8	9
3	6	8	X	10
4	7	9	10	X

Here, with this design paired comparison experiments can be conducted in two ways.

(*a*) We run pairs of treatments for those treatments lying in the same column *i.e.* (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (1, 6), (1, 7), (5, 6), (5, 7), (6, 7), (2, 5), (2, 8), (2, 9), (5, 8), (5, 9), (8, 9), (3, 6), (3, 8), (3, 10), (6, 8), (6, 10), (8, 10), (4, 7), (4, 9), (4, 10), (7, 9), (7, 10) and (9,10). Out of the 45 comparisons possible $\frac{2}{3}$rd are selected by the design.

(*b*) For $n>4$ *i.e.* when the size of the square is >4, we run pairs of treatments for those treatments not lying in the same column. For the above design the pairs are (1, 8), (1, 9), (1, 10), (2, 6), (2, 7), (2, 10), (3, 5), (3, 7), (3, 9), (4, 5), (4, 6), (4, 8), (5, 10), (6, 9), (7, 8). Dykstra (1960) has discussed the precision of these designs in the null case for pairs that have been run and pairs not run. He has discussed these designs in the light of the use of balanced designs for paired comparisons. He has also treated the use of square designs for unequal repetitions in paired comparisons. The designs are constructed by placing the t treatment numbers into a square of size s and running pairs of treatments common to a column or row. For $t=9$, we have

1	2	3
4	5	6
7	8	9

We run pairs among (1, 4, 7), (2, 5, 8), (3, 6, 9), (1, 2, 3), (4, 5, 6) and (7, 8, 9). In another way of forming pairs, we pair each treatment with those treatments not lying in the same row or columns of the association scheme. Cyclic designs of Clatworthy can also be used for pairwise comparison. These designs can be used for fractionating pairs.

Thus if we have n objects, the group of paired comparisons can be split up into subgroups one consisting of $(n-1)$ pairs containing a stipulated object and the other containing

$$\frac{n(n-1)}{2}-(n-1)=\frac{(n-1)(n-2)}{2}$$

pairs. Comparison of one object is completely lost in the second group whereas all comparisons are obtainable from the first group. If $(n-1)$ is a perfect square, square designs of the type above can be used for our comparison. If the design is replicated three times as shown below for $n=10$, we can eliminate judge to judge and time to time variations.

Judge No.	*I*	*II*	*III*
1	1 2 3	4 5 6	7 8 9
2	4 5 6	7 8 9	1 2 3
3	7 8 9	1 2 3	4 5 6

Time variations here can be obtained as replicate comparisons and judge variations as between rows. The randomisations can be restricted to within replications and within judges. The order of presentation of the pairs (X, Y) or (Y, X) can be fixed at random.

Triads can also be split up on the basis suggested above. With 5 objects, the number of triads containing a specified object is the same as the number of pairs that we can form out of the 4 other objects, that is 6. When the result is generalised to n objects, the number of triads that we obtain in the fraction of our choice would be $(n-1)C_2$. For five objects A, B, C, D, E, the fraction of our choice is ABC, ABD, ABE, ACD, ACE, ADE, where A is the specified object. The other group which consists of

$$\frac{(n-1)(n-2)(n-3)}{6}=4$$

triads will not contain comparisons with A and is hence not of interest in case A is important. In other cases that group can also be used for experimental purposes.

We can generalise this procedure even to tetrads. If we consider 6 objects A, B, C, D, E, F, the number of tetrads is $6C_4=15$. The number of tetrads which contain a specified object among n objects is $(n-1)C_3$ which equals 10 in this case. This group will give all the required comparisons and ranking can proceed on this basis. Designs for these cases as well as for complete triads are discussed subsequently.

Another Design For Fractional Pairs

Sadasivan (1967) has investigated the applicability of the latinised rectangular lattice of Harshbangher and Davis for paired comparison experiments. Further investigations showed that it can be used for experiments in fractional pairs as well. If we select all the pairs not containing A out of the objects A, B, C, D, E, and number them as

1	2	3	4	5	6
BC	*BD*	*BE*	*CD*	*CE*	*DE*
7	8	9	10	11	12

and use the rectangular lattice for 12 treatments using sets as judges and rows as time, the plan will be as shown below :—

	Set. I	*Set. II*	*Set. III*	*Set. IV*
Row (1)	(*BC*) (*BD*) (*BE*) 1 2 3	(*CD*) (*BC*) (*CD*) 4 7 10	(*CE*) (*BE*) (*CE*) 5 9 11	(*DE*) (*BD*) (*DE*) 6 8 12
Row (2)	(*CD*) (*DE*) (*CE*) 4 6 5	(*BC*) (*BD*) (*CE*) 1 8 11	(*BE*) (*BC*) (*DE*) 3 7 12	(*BD*) (*BE*) (*CD*) 2 9 10
Row (3)	(*BE*) (*BC*) (*BD*) 9 7 8	(*BD*) (*CE*) (*DE*) 2 5 12	(*BC*) (*DE*) (*CD*) 1 6 10	(*BE*) (*CD*) (*CE*) 3 4 11
Row (4)	(*DE*) (*CE*) (*CD*) 12 11 10	(*BE*) (*DE*) (*BE*) 3 6 9	(*BD*) (*CD*) (*BD*) 2 4 8	(*BC*) (*CE*) (*CE*) 1 5 7

Here a separate randomisation can be performed for each group. Moreover since each pair is repeated twice in a row, it can be presented to the judges in the order (X, Y) in one case and (Y, X) in the other. This allocation can be made in a random sequence. A simple latin square can also be used for comparison of fractional pairs. Associating rows with time and columns with judges and using the fraction containing A in all the pairs from A, B, C, D, E we get the arrangement given below :

		Judge		
Time	*I*	*II*	*III*	*IV*
(*i*)	*AB*	*AC*	*AD*	*AE*
(*ii*)	*AC*	*AD*	*AE*	*AB*
(*iii*)	*AD*	*AE*	*AB*	*AC*
(*iv*)	*AE*	*AB*	*AC*	*AD*

Here also the pairs can be offered in the sequence (X, Y) or (Y, X) at random. Similar designs can be used for fractional triads as well.

Paired comparisons can also be made by using linked designs. Designs of an optimum kind which balance number of comparisons, objects compared, number of observers on given comparisons etc. are probably very rare. If something is to be sacrificed, it is to be done according to the relative importance of the factor. Kendall (1955) has defined the preference polygon and the tours that can be preferred in this polygon from one vertex to another for the objects A, B, C, D, E, F, G. The different tours of the preference polygon are

A	B	C	D	E	F	G
A	C	E	G	B	D	F
A	D	G	C	F	B	E

Each tour involves seven comparisons and each object is compared with two others in a tour. The comparisons in the first case are (A, B); (B, C); (C, D); (D, E); (E, F); (F, G) and (G, A). All the 21 distinct pairs are got from the tours. For a complete set of comparisons each observer would have to make 21 comparisons. But since this may be wearying to the judges we may allocate 14 consisting of 2 tours to each observer. Suppose the tours are represented by a, b, c and observers are 1, 2, 3, the design will take the following form

1	a,	b	
2	b,	c	(H)
3	c,	a	

Here each tour is made equally often and each pair is replicated twice and each observer makes the same number of comparisons. Every observer has a tour in common with every other observer. Thus every observer can be compared with every other observer in respect of two comparisons involving any specified object. If more observers are there we can replicate the design for any number of observers which is a multiple of 3. If we have 11 objects there are five distinct tours round the preference polygon. If the objects be designated A to K, the tours are

a:	A	B	C	D	E	F	G	H	I	J	K
b:	A	C	E	G	I	K	B	D	F	H	I
c:	A	D	G	J	B	E	H	K	C	F	I
d:	A	E	I	B	F	J	C	G	K	D	H
e:	A	F	K	E	J	D	I	C	H	B	G

To preserve balance we have to allot 4 tours to each observer 1, 2, 3, 4, 5

1 :	*b*,	*c*,	*d*,	*e*
2 :	*c*,	*d*,	*e*,	*a*
3 :	*d*,	*e*,	*a*,	*b*
4 :	*e*,	*a*,	*b*,	*c*
5 :	*a*,	*b*,	*c*,	*d*

Here each tour goes to 4 observers and each observer makes 44 comparisons. Comparisons between observers is also possible here. But variations due to time are not taken into account. If we sacrifice symmetry and cut the size of the design we can use a plan as below:—

1 :	*a*,	*b*	
2 :	*b*,	*c*	
3 :	*c*,	*d*	(*A*)
4 :	*d*,	*e*	
5 :	*e*,	*a*	

Here every observer can be compared with two other observers, but not every pair can be compared. If we can have 10 observers we can give each pair of tour to every observer. Then each observer will be making 22 comparisons. Each observer can then be compared with four other observers. We can also use a linked design in the present case. Number the pairs from 1 through 55. With 11 observers and 10 preferences for each, the design is

Ob :	1 :	1	2	3	4	5	6	7	8	9	10
	2 :	1	11	12	13	14	15	16	17	18	19
	3 :	2	11	20	21	22	23	24	25	26	27
	4 :	3	12	20	28	29	30	31	32	33	34
	5 :	4	13	21	28	35	36	37	38	39	40
	6 :	5	14	22	29	35	41	42	43	44	45
	7 :	6	15	23	30	36	41	46	47	48	49
	8 :	7	16	24	31	37	42	46	50	51	52
	9 :	8	17	25	32	38	43	47	50	53	54
	10 :	9	18	26	33	39	44	48	51	53	55
	11 :	10	19	27	34	40	45	49	52	54	55

Here every judge is compared with every other judge by means of only one comparison and every pair is compared twice. The randomisation can be performed within observers and the pairs offered in one of the unknown random sequences (*X*, *Y*) or (*Y*, *X*).

These types of designs can be adapted for fractional pairs as well when the number in the fraction is large. In the case of 6 objects *A* to *F* our fraction consists of the five pairs *AB*, *AC*, *AD*, *AE*, *AF*. Since the other fraction is the complete group of paired comparisons of the object *B* to *F* the terms would be

a :	*B*	*C*	*D*	*E*	*F*
b :	*B*	*D*	*F*	*C*	*E*
c :	*B*	*E*	*C*	*F*	*D*

and hence the terms can be tried in balanced designs of the above type (*H*) or a latin square. But if we select the first fraction which consists of the pairs *AB*, *AC*, *AD*, *AE*, *AF* a linked design of the types given below or a latin square can be used.

Design (1) Ob :		Design (2) Ob :		Design (3) Ob :	Time 1	2	3	4	5	
1	*AB AC*	1	*AB AC AD*	1	*AB*	*AC*	*AD*	*AE*	*AF*	
2	*AC AD*	2	*AC AD AE*	2	*AC*	*AD*	*AE*	*AF*	*AB*	(*B*)
3	*AD AE*	3	*AD AE AF*	3	*AD*	*AE*	*AF*	*AB*	*AC*	
4	*AE AF*	4	*AE AF AB*	4	*AE*	*AF*	*AB*	*AC*	*AD*	
5	*AF AB*	5	*AB AC AD*	5	*AF*	*AB*	*AC*	*AD*	*AE*	

In design (1) the observers can be compared in pairs in a cyclic sequence. Every pair is replicated twice. The pairs can be offered to the observers in the sequence (*X*, *Y*) and (*Y*, *X*) the choice being made at random. In design (2) the observers 1 & 2 can be compared by two pairs whereas observers 1 & 3 by one pair. Every pair is compared thrice. In the latin square, observer as well as time variation can be studied.

When the fraction itself is very large we can try balanced designs or linked designs. With n objects our desirable fraction contains $n-1$ pairs. If we have 22 varieties a linked design with 7 observers and 6 pairs per observer can be tried. Designating the pairs by 1 to 21 the design would take the following form

Ob :	1 :	1	2	3	4	5	6	
	2 :	1	7	8	9	10	11	
	3 :	2	7	12	13	14	15	(*C*)
	4 :	3	8	12	16	17	18	
	5 :	4	9	13	16	19	20	
	6 :	5	10	14	17	19	21	
	7 :	6	11	15	18	20	21	

Here every pair is compared twice. We can set observer comparison also in pairs. But time variations cannot be accounted for.

Let us now proceed to discuss the designs suitable for triad comparisons. When preferences are expressed for a triad A, B, C it may, say, take the form $A>B>C$ *i e.* A is preferred to B and B is preferred to C. The arrangement expresses three preferences $A>B$, $B>C$ and $A>C$ The preference table in such a case would contain $n \times k\, c_3 \times 3$, preferences where n is the number of judges and k, the number of objects. When k is large we can use samples from triads or fractional triads as proposed above. The problems in sampling from triads will be discussed elsewhere. Incomplete block designs can be used for triad comparisons as well. Thus if we have 6 objects to be tested with parameters $v=6$, $b=10$, $k=3$, $r=5$ and $\lambda=2$ and $E=.80$ where E is the efficiency factor of the 20 triads possible only 10 are used in the design. These can be selected at random. The plan is given below :

Judge			
(1)	1	2	5
(2)	1	2	6
(3)	1	3	4
(4)	1	3	6
(5)	1	4	5
(6)	2	3	4
(7)	2	3	5
(8)	2	4	6
(9)	3	5	6
(10)	4	5	6

In another type of plan the varietal positions in the design can be taken by the individual triads. With 3, 4, 5 & 6 things we can form

1, 4, 10 and 20 triads. With 10 triads we can use incomplete block designs as given by the plans 11·14, 11·15, 11·16, 11·17 or 11·18 in Cochran and Cox. To use plan 11·15 we can serially number the triads from 1 to 10 and allocate the triads to the varietal positions in the design. The usual preference matrix can be prepared from the results of such a trial for each triad. Such preference tables can be pooled over all triads. From the proportion of preferences from the whole design the varieties can be ranked according to merit. The estimate of proportion of preferences that we get by this method will be balanced over the judges. If these proportions are obtained after proper randomisation their order can be tested by Chassan's test for order (Sadasivan 1967).

It is further found that by using a latin square design we can eliminate judge to judge and time to time variation. This design can be used for triads from 4 or 5 objects. The analysis can be performed from such an experiment as well by forming the preference matrix and testing as in the previous case. Other models for analysis are briefly discussed subseqently. The order of presentation in the case of triads are (X, Y, Z), (X, Z, Y), (Y, Z, X), (Y, X, Z), (Z, X, Y), (Z, Y, X). Any one of these orders can be chosen at random and the objects presented to the judges.

The latinised rectangular lattice (Harshbangher and Davis 1952) can also be used for triad comparisons. For using the design the numbers of triads must be expressible as $K(K-1)$ where K is an integer. Thus with n objects we have to solve the equation

$$K(K-1)=n(n-1)(n-2)/6$$

which reduces to

$$6K^2-6K-(n^3-3n^2+2n)=0.$$

Solving for K we get

$$K=\tfrac{1}{2}\pm\tfrac{1}{6}\sqrt{9+6(n^3-3n^2+2n)}$$

Trying $K=3$ this leads to the cubic

$$n^3-3n^2+2n-36=0$$

which shows that the design for $K=3$ cannot be used for triads. Trying $K=4, 5$, etc. we find that no simple solution exists. Thus for n objects we can try a rectangular lattice with $n(n-1)$ triads in a rectangle. If it is replicated $(n-2)/6$ times our conditions will be satisfied. For $n=4$, only 1/3rd the replicate is required to accommodate the 4 triads.

Hence the design for 4×3 varieties can be used for the 4 triads by allocating the triads at random as below :

1	2	3	4	5	6	7	8	9	10	11	12
ABD	*BCD*	*ABC*	*ACD*	*ABD*	*ABC*	*ACD*	*BCD*	*ABD*	*ABC*	*ACD*	*BCD*

	Judge I			*Judge II*			*Judge III*			*Judge IV*		
Time (1)	1	2	3	4	7	10	5	9	11	6	8	12
,, (2)	4	6	5	1	8	11	3	7	12	2	9	10
,, (3)	9	7	8	2	5	12	1	6	10	3	4	11
,, (4)	12	11	10	3	6	9	2	4	8	1	5	7

Here each judge tries each triad 3 times. Each triad is tried thrice on each occasion. The interactions of time and judges can also be found from the design.

THE EXTENT OF REDUCTION BY FRACTIONATION

The percentage of reduction by fractional pairs for n objects is seen to be $\frac{100(n-2)}{n}$ and that by fractional triads is $\frac{100(n-3)}{n}$. Since the former is greater than the latter, the percentage reduction attained by fractionating pairs is more than that attained by fractionating triads. The reduction attained in tetrads is $\frac{100(n-4)}{n}$ and is still less. The trend continues for other higher order preference comparisons. In this sense fractionating pairs is more efficient than fractionating triads.

Again, as n increases the relative gain in percentage reduction by fractional pairs decreases whereas the percentage reduction increases in absolute value. For $n=2$, the percentage reduction achieved is nil. After $n=6$ the percentage reduction achieved for each increase of unity for the value of n is less than 5. Further the relative gain in percentage reduction is less than 5 when $n=8$ or more. This is seen by tabulating the values of $\frac{dP}{dn}$ and $\frac{1}{P}\frac{dP}{dn}$ where $P=\frac{100(n-2)}{n}$ as given below :

n	P	$\frac{1}{P}\frac{dP}{dn}$
2	0	∞
3	33·3	66·7
4	50·0	25·0
5	60·0	13·3

6	66·7	8 0
7	71·4	5·7
8	75·0	4·2
9	77·8	3·1
10	80 0	2·5

The efficiencies of the alternative designs which can be used for pairs, triads or fractions thereof are under investigation.

Methods Of Analysis

Data on preferences can be analysed by (1) Bradley-Terry Model (2) Thurstone-Mosteller Model, (3) Combinatorial method. The first model and a variant of the second model were discribed at the outset. The Bradely-Terry model can be directly adapted for fractional pairs where each judge is examining each pair an equal number of times. But in partially balanced designs, the model will have to be modified. How the Thurstone-Mosteller Model modifies for fractional pairs remains to be examined. The Combinatorial method of analysis can also be used in all the designs discussed above. The technique in certain specific cases are detailed below.

For designs of the linked type (A) the preference data can be formed and the proportions tested. The preference table from the latin square type of design (B) is better since the size of the experiment is larger. The test is Chassan's test for order as examined by Sadasivan (1965); for a linked design of the type (C) also the preference matrix can be formed and the above test carried out. For designs using different combinations of tours as well, such comparisons and tests can be made. From designs for triad comparisons as well we can use the Combinatorial method of analysis and test.

Summary And Conclusions

Thus when the size of experiments by paired or triad comparisons is large we can reduce the size by using fractional pairs and fractional triads. The designs for such situations as well as those for complete pairs and triads are discussed here. The methods of analysis possible are also indicated.

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